NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

19th March, 1981

Time allowed - 3½ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. H is the orthocentre of triangle ABC. The midpoints of BC, CA, AB are A', B', C' respectively. A circle with centre H cuts the sides of triangle $A'B'C'$ (produced if necessary) in six points, D_1, D_2 on $B'C'$, E_1, E_2 on $C'A'$ and F_1, F_2 on $A'B'$.

Prove that
$$AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2 .$$

2. m and n are positive integers. S_m is the sum of m terms of
$$(n+1) - (n+1)(n+3) + (n+1)(n+2)(n+4) - (n+1)(n+2)(n+3)(n+5) + \ldots$$
where the terms alternate in sign and each, after the first, is the product of consecutive integers with the last but one omitted.

Prove that S_m is divisible by $m!$ but not necessarily by $m!(n+1)$.

3. a, b, c are positive numbers. Prove

(i) $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$.

(ii) $abc \geq (a+b-c)(b+c-a)(c+a-b)$.

4. n points are given such that no plane passes through four of them. S is the set of all tetrahedra whose vertices are 4 of the n points. A plane does not pass through any of the n points.

Prove that it cannot cut more than $n^2(n-2)^2/64$ of the tetrahedra of S in quadrilateral cross-sections.

5. Find, with proof, the smallest possible value of $|12^m - 5^n|$, where m and n are positive integers.

6. a_i, $i = 1,2,3,\ldots n$, are distinct non-zero integers.
$$p_i = \prod_{j \neq i}^{n} (a_i - a_j)$$
is the product of the $(n-1)$ factors $(a_i - a_1)$, $(a_i - a_2)$, ... $(a_i - a_n)$, the zero factor $(a_i - a_i)$ being excluded.

Prove that if k is a non-negative integer,
$$\sum_{i=1}^{n} \frac{a_i^k}{p_i}$$
is an integer.